1.0 ‖‖ 2.8 ‖ 2.5

3.2 ‖ 2.2

3.6

1.1 4.0 ‖ 2.0

1.8

1.25 ‖ 1.4 ‖ 1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

AD-A174 953

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY NA		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release, Distribution Unlimited	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE NA			
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 86-2138	
6a. NAME OF PERFORMING ORGANIZATION Northwestern University	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NM	
6c. ADDRESS (City, State and ZIP Code) Evanston, IL 60601		7b. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC 20332-6448	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR	8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER 82-0189 AFOSR	

8c. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC 20332	10. SOURCE OF FUNDING NOS.			
	PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2304	TASK NO. A5	WORK UNIT NO.

11. TITLE (Include Security Classification)
Reliability of Complex Devices in Random Environments

12. PERSONAL AUTHOR(S)
E. CINLAR and S. OZEKICI

13a. TYPE OF REPORT Interim	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Yr., Mo., Day) July, 1986	15. PAGE COUNT 34

16. SUPPLEMENTARY NOTATION

17.	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Reliability, multi-component devices, random environments
			semimarkov processes.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The lifetimes of the components of a device depend on each other because of their joint dependence on the environmental conditions. We introduce intrinsic age processes, one for each component, to handle such dependence. The data required can be obtained by experiments under controlled laboratory conditions. The computations needed for randomly varying conditions are recursive and can be used for making decisions regarding maintenance and replacement.

DTIC
ELECTE
DEC 1 2 1986
S   E

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5027	22c. OFFICE SYMBOL AFOSR/NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

RELIABILITY OF COMPLEX DEVICES IN RANDOM ENVIRONMENTS*

by

E. ÇINLAR and S. ÖZEKİCİ

Accession For
NTIS GRA&I
DTIC TAB
Unannounced
Justification
By
Distribution/
Availability Codes
Dist   Avail and/or special
A-1

Princeton University
Department of Civil Engineering
Princeton, New Jersey 08544

Engineering-Economic Series
Report EES-86-14

July, 1986

RELIABILITY OF COMPLEX DEVICES IN RANDOM ENVIRONMENTS*

by

E. ÇINLAR and S. ÖZEKİCİ

Consider a device like a jet engine, which is composed of 100 to 150 components. Each component itself is made of many parts, but for purposes of repair and replacement the components are indivisible units. Reliability studies for such a device are made difficult by the complexity of the device and the changing environmental conditions it is subjected to.

The source of difficulty is the stochastic dependence of the components' lifetimes on each other. This dependence is largely caused by the fact that the same environmental conditions affect all components simultaneously.

Moreover, the environmental conditions in the field vary randomly over time and are different from the test environments used in laboratories. A paper by WINTERBOTTOM (1984) reviews some of the recent developments on this issue and highlights the difficulties involved.

In MASTRAN and SINGPURWALLA (1978) and SHAKED (1977) attemps are made to confront the problem by assuming that the lifetimes are "associated" or "positively dependent by mixture." In LINDLEY and SINGPURWALLA (1984), for the case of two components, the lifetimes are assumed to be conditionally independent given a certain parameter, which parameter is then randomized. Thus, the random parameter represents the effect of environment, which is construed to be unchanging over time.

Explicit consideration of the environment as a stochastic process has been attempted in a number of papers. Most such papers take the model of ESARY, MARSHALL, and PROSCHAN (1973) as the starting point, and model the shock and wear "intensities" as randomly varying in response to environmental conditions; see A-HAMEED and PROSCHAN (1973), (1975), ÇINLAR (1984) and FELDMAN (1976), (1977). These models are, however, for one component, and tend to be statistically intractable. The use of multi-components are discussed by ESARY and MARSHALL (1974), and very sophisticated models are introduced in ÇINLAR (1977). The last paper models the environment as an abstract Markov process, lets the Markov process govern the intensities of shock and wear for each component, and view the deterioration levels of components as an n-dimensional process with conditionally independent increments given the intensity processes. However, the mathematical complexity and statistical intractability of this paper detract much from its usefulness.

In the present paper we introduce a relatively simple model. Environment process is explicitly considered, but its effect on the components is made deterministic. For this purpose, we introduce a concept, which we call intrinsic age, in order to relate the deterioration of a component under field conditions to the deterioration it would have experienced under laboratory conditions. The data required are the distribution functions $t \to F_k(x,t)$, one for each component k and each environmental state x, where $F_k(x,t)$ is the probability that the lifetime of the component k is at most t given that the environmental state remains fixed at x throughout.

We envision these distribution functions to be obtained, by statistical and engineering considerations, from test data in laboratory. Afterward, based on these and the probabilistic structure of the environment process, certain rules for maintenance and replacement decisions would be worked out by mathematical means. For the operating engineer who is to make those decisions daily, we provide simple schemes to compute the quantities needed for decision making. Although we shall develop the concept fully, it seems particularly useful if the number of environmental states can be kept small; for instance, for the jet engine alluded to above, it would be helpful if the environmental states were summarized as take-off, cruise, landing, and idleness, instead of the exact descriptions provided by temperatures, accelerations, vibrations, and so on.

The paper is organized as follows. In the first section, the basic idea is introduced in the simple case of one component and in a non-technical style. The basic model is described in the second section. Some reliability computations are discussed, in the final section, assuming that the environment evolves as a semimarkov process; these computations would be needed to determine optimal policies for maintenance and replacement, which issues are not touched in this paper. Finally, in the appendix a brief account is given of semimarkov processes and related results.

1. INTRINSIC AGE OF A COMPONENT

Our aim in this section is to introduce the basic idea in the simple case of one component and in a non-technical style.

Consider one component. Suppose that, throughout its lifetime, the environmental conditions remain fixed at x. Let $H(x,t)$ be the cumulative hazard function in t under this condition, that is, if the probability that the component's lifetime exceeds t is $1 - F(x,t)$, then

$$(1.1) \qquad 1 - F(x,t) = \exp(-H(x,t)) , \qquad t \geq 0 .$$

Observe that this is also the probability that $\hat{L}$ exceeds $H(x,t)$ where $\hat{L}$ is an exponentially distributed random variable with mean one.

We view $H(x,t)$ as the <u>intrinsic age</u> of the component at time t given that the environmental state remains x throughout $[0,t]$ and that the component was new to start with. Then, $\hat{L}$ may be thought as the <u>intrinsic lifetime</u>: when the intrinsic age reaches the intrinsic lifetime, the component fails.

Assuming that the distribution function $t \rightarrow F(x,t)$ has a density, so does the increasing function $t \rightarrow H(x,t)$, and the derivative of the latter is called the hazard rate function. In this case, it is profitable to introduce a new function, which we call the <u>intrinsic ageing rate</u> function, by

$$(1.2) \qquad r(x,a) = \frac{d}{dt} H(x,t)\Big|_{t=\tau(x,a)} , \qquad a \geq 0 ,$$

where $\tau(x,a)$ is the time at which the intrinsic age becomes a:

(1.3) $$\tau(x,a) = \inf \{t: \; H(x,t) > a\} \; .$$
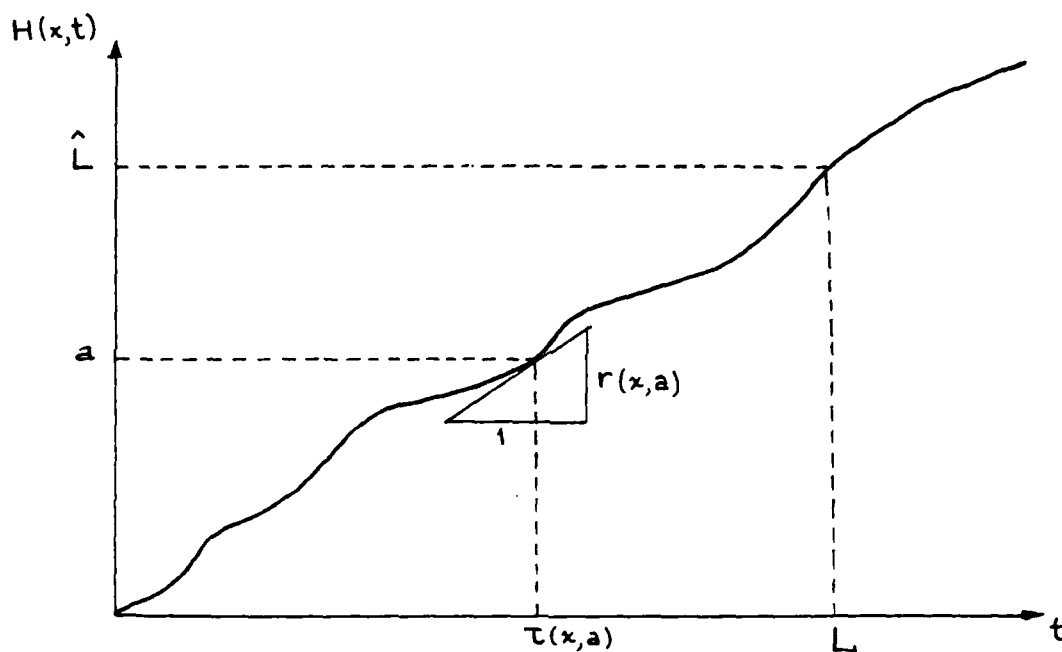
These relationships are pictured in Figure 1 below.

Figure 1. Evolution of intrinsic age under a fixed environmental state x. Here, $\hat{L}$ is the intrinsic lifetime and L is the real lifetime.

We come now to the main point. Suppose that the component is subject to varying environmental conditions. Our basic assumption is that, at any time t, the rate (per unit time) of increase of intrinsic age is $r(x,a)$ if the environment is in state x and the intrinsic age is a, this being independent

of how the environment and intrinsic age have arrived at their present

values x and a. In other words, letting X_t and A_t denote the environmental

state and intrinsic age respectively at time t, we assume that

(1.4) $$dA_t = r(X_t, A_t) \, dt \, , \qquad t > 0 \, .$$

Finally, as before, we assume that the intrinsic lifetime $\hat{L}$ is a standard

(mean one) exponential random variable independent of the environmental process

(X_t), and we model the lifetime L of the component as the time at which

intrinsic age is about to exceed $\hat{L}$:

(1.5) $$L = \inf \; \{t : A_t > \hat{L}\} \, .$$

By the continuity of $t \to A_t$ implicit in the main assumption, we indeed have

$\hat{L} = A_L$.

Next, we describe various implications of the model that (1.4) and (1.5)

sets up. We do this in plain language, call the results "principles," and

omit proofs. The first statement is a re-statement of (1.1) - (1.2): if (A_t)

satisfies (1.4) then r figuring in (1.4) is as described earlier by (1.1) - (1.2).

The remaining statements show the workings of the model in the simple case where

the environmental process (X_t) is piecewise constant. They suggest an easy

updating scheme for the intrinsic age process, which would be used in practice

for making decisions on when to do maintenance.

(1.6) IDENTIFICATION PRINCIPLE. Suppose that the component is new at the beginning and spends all its life in a fixed environmental state x. Let $t \to F(x,t)$ be the distribution function of the lifetime under this condition. Then, the intrinsic age of the component at time t is

$$A_t = H(x,t) = -\log\left[1 - F(x,t)\right] .$$

(1.7) AGEING PRINCIPLE. Suppose that a mission is to start at time s and last t time units, and the environmental state is to remain x throughout the mission. If the intrinsic age at the start is $A_s = a$, then at the end it is

$$A_{s+t} = h(x,a,t) = H(x,\tau(x,a) + t) .$$

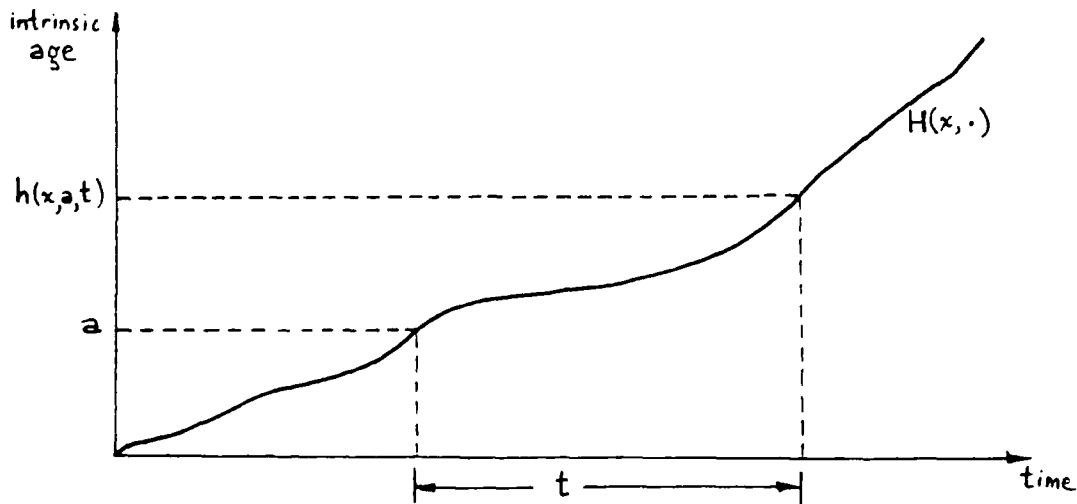(See Figure 2 below.) We call $h(x,a,t) - a$ the amount of ageing caused by the mission.

Figure 2. If the intrinsic age is a at the beginning, then it is h(x,a,t) after t time units spent under condition x. Generally, $\tau(x,a)$ is not equal to the starting time s.

(1.8) ADDITIVITY PRINCIPLE. Suppose that a mission consists of n sub-missions causing $a_1, \ldots, a_n$ amounts of ageing. Then, the amount of ageing caused by the mission is $a_1 + \ldots + a_n$.

(1.9) RELIABILITY PRINCIPLE. Suppose that a mission is to cause an amount a of ageing. Given that the component has not failed before the mission starts, its reliability in the mission is e^{-a}.

We illustrate these principles with an example. For simplicity, we consider a mission with two stages, during which the environmental states are x and y. Let $H(x,t)$ and $H(y,t)$ be given according to the identification principle (1.6). Initially, when the mission starts at time s, the component's intrinsic age is known to be a. The component spends t time units under the environmental state x and u time units under y. According to the ageing principle, then, its intrinsic age will be $b = h(x,a,t)$ at time s+t and $c = h(y,b,u)$ at time s+t+u. See Figure 3 below. So, the mission is to cause c-a amount of ageing, and the reliability is $e^{-(c-a)}$.
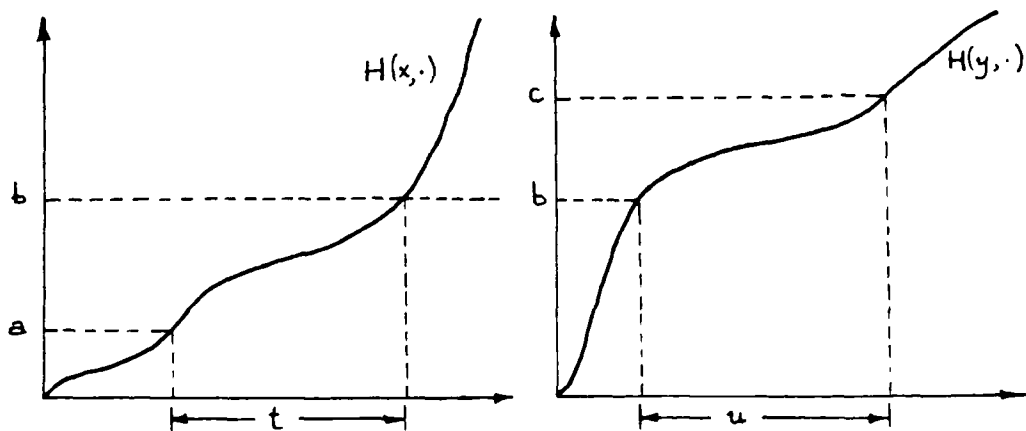

Figure 3. Starting with intrinsic age a, t time units spent under x makes it b and u time units spent under y makes it c.

As the example shows, the ageing principle is in fact an updating scheme assuming that the environmental state is known. In practice, as time flows, past history of the environmental process will become known and the intrinsic age can be computed by the principles here. The decisions regarding preventive maintenance and replacement would then be made on the basis of the computed intrinsic age. On the other hand, finding the optimal maintenance and/or replacement policies require computations of a different sort: then the future has to be take. into account, which requires a stochastic description of the environmental process (X_t). We shall take up computations of that kind in Section 3.

2. DEVICES WITH MANY COMPONENTS

In this section we describe the basic model for a device with many components.

Throughout, $(\Omega, \mathbf{A}, P)$ will be a complete probability space. We write $\mathbf{R}_+$ for $[0, \infty)$. To save on notation, we do not introduce σ-algebras explicitly on $\mathbf{R}_+$, and later, on other spaces.

Let E be a set equipped with a σ-algebra. Elements of E are called *environmental states* (or *states*, simply). Elements of the σ-algebra on E are called the measurable subsets of E. We suppose that, for each state x in E, the singleton $\{x\}$ is measurable. We distinguish a point in E, denote it by δ, and think of it as the state that causes no ageing (idle state for the device). We let $X = (X_t)_{t \in \mathbf{R}_+}$ be a stochastic process with state space E; it represents the environmental process. For the present, we leave its probability law unspecified.

We let K denote the collection of all components. It is a finite set, its elements are called components. For each k in K, we let $\hat{L}(k)$ be an exponentially distributed random variable with mean one; it stands for the intrinsic lifetime of the component k. We assume that $\hat{L}(k)$, $k \varepsilon K$, are independent of each other and of the process X.

We let F denote the collection of all positive vectors $a = (a(k))_{k \varepsilon K}$, that is, $F = \mathbf{R}_+^K$. Each a in F represents a vector of intrinsic ages of all components. We let $A = (A_t)_{t \varepsilon \mathbf{R}_+}$ be an increasing continuous stochastic process taking values in F; that is, for each component k, $t \to A_t(k)$ is increasing and continuous and takes values in $\mathbf{R}_+$; it represents the intrinsic age process for component k.

We define the lifetime of component k by

(2.1) $$L(k) = \inf \{t: \ A_t(k) > L(k)\} \ ,$$

and let $L = (L(k))_{k \in K}$ be the vector of lifetimes. The following is the main assumption regarding the structure of A.

(2.2) HYPOTHESES. i) For each $k \in K$ there exists a positive measurable function r_k on $E \times F$ such that

(2.3) $$dA_t(k) = r_k(X_t, A_t)dt \ , \qquad t > 0 \ .$$

(ii) We have $r_k(x,a)$ strictly positive for each $k \in K$, $a \in F$, and $x \in E$ with $x \neq \delta$. For $x = \delta$, the idle state in E, we have $r_k(x,a) = 0$ for all $k \in K$ and $a \in F$.

The basic hypothesis is the first one: the intrinsic age process A is a deterministic functional of the environment process X. In particular, (2.3) implies that the rate of increase of the intrinsic age of component k at time t is a function of the identity of that component, the present state of the environment, and the present intrinsic ages of all the components. Hypothesis (ii) is a regularity condition that is meant to ensure that (2.3) has a unique solution; in particular, it singles out the state δ as the only state that causes no ageing.

A special case of interest is when $r_k(x,a)$ depends on $a \in F = \mathbb{R}_+^K$ only through $a(k)$, that is,

(2.4)
$$r_k(x,a) = \hat{r}_k(x,a(k)) \quad \text{if} \quad a = (a(k))_{k \in K} \ .$$

In this case, the intrinsic ages $A_t(k)$, $k \in K$, do not interact except through their common dependence on X. Then, the results of the preceding section applies to each component separately. This case is important in applications because of its computational simplicity. But we do not assume this simpler case because it does not reduce the mathematical development appreciably.

In the further special case where $r_k(x,a)$ is free of a, say,

(2.5)
$$r_k(x,a) = \tilde{r}_k(x) \ ,$$

we have

$$A_t(k) = A_0(k) + \int_0^t \tilde{r}_k(X_s) \, ds \ ,$$

and $t \rightarrow A_t(k)$ is an additive functional of X. This case has been studied in some depth, see PARKUS and BARGMAN (1970) and especially ÇINLAR (1977) where this case occurs as a special case of a model different from the general one here.

In the special case, uninteresting from our present perspective, where $r_k(x,a)$ is free of x, the process A becomes deterministic. Since the intrinsic lifetimes $\hat{L}(k)$ are independent of each other, in this case the lifetimes $L(k)$ are also independent. Note that, though the lifetimes $L(k)$ are

stochastically independent, this case does accommodate deterministic inter-
actions between the intrinsic age processes.

Going back to the general case, we summarize our point of view once
more. Every component is endowed with an intrinsic lifetime that is
independent of everything else and has the exponential distribution with
mean one. The distinguishing characteristics of a component k are summarized
by the function $r_k(x,a)$, which is the response (the ageing rate) of k to the
environmental state x at a time when the intrinsic ages of the components are
$a(j)$, $j \varepsilon K$.

We shall think of A_t as a function defined on K and taking random values
in $\mathbb{R}_+$; similarly, we think of elements $a \varepsilon F$ as functions defined on K and
taking values in $\mathbb{R}_+$. We let $r(x,a)$ denote the function on K whose value at
$k \varepsilon K$ is $r_k(x,a)$. For computational and display purposes, functions on K
ought to be thought as column vectors. We let σ denote the counting measure
on K, it can be thought as a row vector whose entries are all equal to one.
Thus, for $a \varepsilon F$ we have

$$(2.6) \qquad \sigma a = \sum_{k \varepsilon K} a(k) .$$

With these notational conventions, the basic hypothesis expressed by the differential equation (2.3) can be re-stated as

$$(2.7) \qquad dA_t = r(X_t, A_t) \, dt \,, \qquad t > 0 \,.$$

Solution of this can be obtained by mimicking the one-dimensional (one component) case discussed in the preceding section. We describe a few details for the special case where the environment process X is piecewise constant.

For fixed $x \varepsilon E$, let $t \to h(x,a,t)$ denote the solution of

$$(2.8) \qquad df(t) = r(x,f(t)) \, dt, \qquad t > 0 \,,$$

with the initial condition

$$(2.9) \qquad f(0) = a \,.$$

In other words, the k-entry of column vector $h(x,a,t)$ is the intrinsic age of component k at time $s+t$ assuming that the environment remained in state x throughout $[s,s+t)$ and the intrinsic ages of the components at time s were $a(j)$, $j \varepsilon K$. In particular, for $x = \delta$, we have

$$(2.10) \qquad h(\delta,a,t) = a \,, \qquad t \geq 0 \,,$$

because of the assumption that $r(\delta,a) = 0$ for all a.

Next, we describe the construction of the path $t \rightarrow A_t$ as a functional of the trajectory $t \rightarrow X_t$ in the simple and important case where $t \rightarrow X_t$ is piecewise constant. Let $T_0 = 0$ and let $T_1, T_2, \ldots$ be the successive jump times of X. By the term piecewise constant, we mean that $t \rightarrow X_t$ is constant over each interval $[T_n, T_{n+1})$ and that the increasing sequence of times T_n approaches $+\infty$ as n goes to $+\infty$. We let Y_n be the value of X_t for $t \in [T_n, T_{n+1})$ for each integer $n \geq 0$, that is,

$$(2.11) \qquad X_t = Y_n \quad \text{for} \quad T_n \leq t < T_{n+1} .$$

Given A_0, we first define the values B_n of A_t for $t = T_n$ recursively:

$$(2.12) \qquad B_0 = A_0 ; \quad B_{n+1} = h(Y_n, B_n, T_{n+1} - T_n) , \quad n \geq 0 ;$$

and then put

$$(2.13) \qquad A_t = h(Y_n, B_n, t - T_n) \quad \text{for} \quad T_n \leq t < T_{n+1} .$$

This defines A_t for all $t \in \mathbb{R}_+$, and in view of the way h is defined as the solution to (2.8) and (2.9), the resulting path $t \rightarrow A_t$ satisfies (2.7).

3. RELIABILITY CONSIDERATIONS

This section is devoted to various computations on the reliability of
the device. These computations would be needed for purposes of decision
making on maintenance and replacement studies.

Throughout, we assume that the environment process $X = (X_t)_{t \in \mathbb{R}_+}$ is a
piecewise constant semimarkov process with state space E and semimarkov
kernel Q. In other words, with the jump times T_n and the successive states
Y_n defined as in (2.11), we assume that

$$(3.1) \qquad P\left[Y_{n+1} \epsilon C, \ T_{n+1} - T_n \ \epsilon \ D \,|\, Y_0, \ldots, Y_n; \ T_0, \ldots, T_n\right] = Q(Y_n, C, D)$$

for every integer $n \geq 0$, measurable set $C \subset E$, and Borel set $D \subset \mathbb{R}_+$.
For further information on the processes (X_t), (Y_n, T_n), and the semimarkov
kernel Q we refer to the appendix. At this point, we should mention that,
then, (Y_n) is a Markov chain with state space E and transition kernel

$$(3.2) \qquad\qquad P(x, C) = Q(x, C, \mathbb{R}_+) \ ,$$

and that, given the Markov chain (Y_n), the successive sojourn durations
$T_1-T_0, T_2-T_1, \ldots$ are conditionally independent, with the conditional distribu-
tion of $T_{n+1} - T_n$ depending on Y_n and Y_{n+1} only, but otherwise arbitrary.
In the special case where the sojourn distributions are exponential and depend
only on Y_n, that is, if

$$(3.3) \qquad Q(x,C,D) = P(x,C) \int_D \lambda(x) e^{-\lambda(x)t} \, dt \; ,$$

then the semimarkov process X becomes Markov.

In general, the sequence (Y_n, T_n) satisfying (3.1) is called a Markov renewal process with state space E (that the T_n take values in $\mathbf{R}_+$ is understood). Moreover, then, defining $B_n = A_{T_n}$ as in (2.12), the sequence $((Y_n, B_n), T_n)$ is again a Markov renewal process, the Markov chain (Y_n, B_n) taking values in the space $\hat{E} = E \times F$. In fact, in the computations to follow, it is (Y_n, B_n, T_n) that plays the central role. Note that, even if we assumed E to be discrete, the space $\hat{E}$ is not discrete and hence there is little to be gained mathematically by specializing E to be discrete (computations in practice are another matter). However, if E is discrete, the corresponding formulas can be obtained by replacing dy and dz below by y and z and changing the integrals over y and z to summations.

Recall that the lifetimes $L(k)$, intrinsic lifetimes $\hat{L}(k)$, and intrinsic ages $A_t(k)$ define random functions L, $\hat{L}$, and A_t on the finite set K of components. Therefore, $\hat{L} > A_t$ means that $\hat{L}(k) > A_.(k)$ for all $k \in K$, and $L > t$ means that $L(k) > t$ for all $k \in K$. Also, recall that σ is the counting measure on K and acts as the summation operator as described by (2.6).

Finally, we introduce the following notations of convenience. We write 1_D for the indicator function of D, that is, $1_D(x)$ is equal to 1 or 0 according as x is in D or not. We let

$$(3.4) \qquad H(x,a,t;C) = 1_C(h(x,a,t)) \; ,$$

(3.5) $\qquad q(x,t) = Q(x,E,(t,\infty)) = P[T_{n+1} - T_n > t | Y_n = x]$,

and, for all $x \varepsilon E$ and $a \varepsilon F$, we write

(3.6) $\qquad P_{xa}[\cdot] = P[\cdot | X_0 = x, A_0 = a, \mathcal{L} > a]$,

that is, the conditional probability measure given that environment starts in state x, intrinsic ages at a(k), kεE, and all of the components are in working order.

Reliability in a mission of length t

By this we mean the probability that no component fails during $[0,t]$ and that there can be no maintenance, repair, or replacement during $[0,t]$. We compute this probability assuming that the time 0 coincides with a jump of environmental process and that the state x of the environment and the intrinsic ages a(k) of the components are known and it is further known that all components are in working order. Thus, what we want is

(3.7) $\qquad P_{xa}[L > t] = P_{xa}[\mathcal{L} > A_t] = E_{xa} \exp[-\sigma(A_t - a)]$,

in view of (2.1) and the memorylessness of the exponential variables $\mathcal{L}(k)$ and their independence from each other and from X and A.

To this end we use a renewal argument at the time T_1 of first jump for X. If $T_1 > t$, then $A_t = h(x,a,t) = b$ and the probability that no failures occur is $\exp[-\sigma(b-a)]$. If $T_1 = s \le t$, then $A_s = h(x,a,s) = b$, probability

that no failures occur during $[0,s]$ is $\exp[-\sigma(b-a)]$, and the future after s is a probabilistic replica of that at time 0 except that the initial variables now are y and b, where y is the state that X jumps to at time $s = T_1$. Thus,

$$(3.8) \qquad P_{xa}[L > t] = q(x,t) \exp[-\sigma(h(x,a,t) - a)]$$

$$+ \int_{E \times F \times [0,t]} Q(x,dy,ds) \, H(x,a,s; \, db) \, e^{-\sigma(b-a)} P_{yb}[L>t-s] \, .$$

We introduce a semimarkov kernel $\hat{Q}$ on the space $E \times F$ by, in differential form,

$$(3.9) \qquad \hat{Q}(x,a; \, dy,db; \, ds) = Q(x,dy,ds) \, H(x,a,s; \, db) \, e^{-\sigma(b-a)} \, ,$$

where we removed the parentheses in the more correct notation $Q((x,a),d(y,b),ds)$. In the notation of Appendix, the equation (3.8) is a Markov renewal equation: letting $f(x,a,t)$ denote the left side and $g(x,a,t)$ the first term on the right, we have

$$(3.10) \qquad\qquad f = g + \hat{Q} * f \, .$$

According to Proposition (A.14) of Appendix, (3.10) has a unique solution, $f = \hat{R} * g$, where $\hat{R}$ is the Markov renewal kernel corresponding to $\hat{Q}$. Thus, the solution of (3.8) is

$$(3.11) \quad P_{xa}[L>t] = \int_{E \times F \times [0,t]} \hat{R}(x,a; \, dy,db; \, ds) \, q(y,t-s) \exp[-\sigma(h(y,b,t-s) - b)] \, .$$

Perhaps the more important point here is that (3.8) has a unique solution. For, computing $\hat{R}$ requires solving equations of similar difficulty. However, an explicit solution like (3.11) is valuable for theoretical purposes and is useful to those who would seek optimal policies for maintenance. An envisioned use of (3.11) is mentioned in the next paragraph.

Failure before maintenance

At each time T_n of change of state for the environment, the new state Y_n and the intrinsic age vector $B_n = A_{T_n}$ will be known through computations of the sort described in Section 1. We suppose that the times T_n would be the only times at which a maintenance action can be taken, and the decision maker would decide for maintenance by considering only the state Y_n and the vector B_n : if $Y_n = y$ and $B_n = b$, maintenance would be decided for if (y,b) belongs to a certain set M. In other words, the time of first maintenance would be

$$(3.12) \qquad T = \inf \ \{T_n : (Y_n, B_n) \ \epsilon \ M\} \quad ,$$

where M is a pre-specified measurable subset of $E \times F$. The set M itself would be chosen in accordance with further studies considering costs, risks, etc. Of particular importance in such considerations would be probabilities like $P_{xa}[L > t]$, which we just examined, and $P_{xa}[L > T]$, which we shall compute next. For instance, if one or the other or some combination of these two numbers is "too low," then the point (x,a) would belong to M.

Thus, the probability we consider next, $P_{xa}[L > T]$ is the probability that no failures occur before the maintenance time T, given that $X_0 = x$, $A_0 = a$, and no failures occurred before the time taken as 0. We use a renewal argument at the time T_1 of first jump for X, just as in the preceding computation. It yields, for $(x,a) \notin M$,

$$(3.13) \quad P_{xa}[L>T] = \int_{E \times \mathbb{R}_+} Q(x,dy,ds) \, 1_M(y,h(x,a,s)) \, \exp\left[-\sigma(h(x,a,s) - a)\right]$$

$$+ \int_{E \times F \times \mathbb{R}_+} Q(x,dy,ds) \, H(x,a,s; db) \, 1_N(y,b) \, e^{-\sigma(b-a)} \, P_{yb}[L>T] \, ,$$

where we put $N = (E \times F) \backslash M$, the no-maintenance set. This is a linear equation of the form

$$(3.14) \quad f(x,a) = g(x,a) + \int_{E \times F} \bar{P}(x,a; dy,db) \, f(y,b) \, ,$$

and the sub-Markovian kernel $\bar{P}$ is "highly defective":

$$(3.15) \quad \bar{P}(x,a; E,F) < \int_{E \times \mathbb{R}_+} Q(x,dy,ds) \, 1_N(y,h(x,a,s)) \leq Q(x,E,\mathbb{R}_+) = 1$$

for $x \neq \delta$, since $h(x,a,s) = b > a$ for $x \neq \delta$, and for $x = \delta$, it is clear that the next state will differ from δ. It follows that (3.14) has a unique solution, of the form $f = \bar{R}g$ in kernel-function notation, where $\bar{R}$ is the potential kernel corresponding to $\bar{P}$:

$$(3.16) \quad \bar{R}(x,a; dy,db) = \sum_{n=0}^{\infty} \bar{P}^n(x,a; dy,db) \, .$$

Thus, the unique solution of (3.13) is

$$(3.17) \quad P_{xa}[L>T] = \int_{E \times F} \bar{R}(x,a; \, dy,db) \, g(y,b)$$

$$= \int_{E \times F} \bar{R}(x,a; \, dy,db) \int_{E \times \mathbb{R}_+} Q(y,dz,ds) \, \cdot$$

$$1_M(z,h(z,b,s)) \, \exp\left[-\sigma(h(z,b,s) - b)\right] \, .$$

Again, as we remarked after (3.11), this computation shifts the real burden to the computation of $\bar{R}$. On the latter issue, we should note that, in view of (3.15), $\bar{P}^n(x,a; \, E \times F)$ has a geometric tail in n, and approximating $\bar{R}$ should be easy.

At the time of first failure or maintenance

Let T be the time of first maintenance as described earlier, by (3.12). Let S be T or the time of first failure (of some component), whichever is smaller:

$$(3.18) \qquad\qquad S = T \wedge \inf_{k \in K} L(k) \, .$$

In this paragraph we are interested in the time S and the states of environment and intrinsic ages at the time S. Our aim is to compute the following joint "distributions":

(3.19) $Q_M(x,a; D,B; t) = P_{xa}[X_S \epsilon D, A_S \epsilon B, S \le t, S = T]$

(3.20) $Q_k(x,a; D,B; t) = P_{xa}[X_S \epsilon D, A_S \epsilon B, S \le t, \quad S = L(k)]$,

where D is a measurable subset of E, B is a Borel subset of $F = \mathbf{R}_+^K$, and

$t \epsilon \mathbf{R}_+$. Of course,

(3.21) $Q_M(x,a; D,B; t) + \sum_{k \epsilon K} Q_k(x,a; D,B; t) = P_{xa}[X_S \epsilon D, A_S \epsilon B, S \le t]$.

These distributions would be needed in studies concerning optimal maintenance
and replacement and in figuring out system performance characteristics in the
long run.

These computations are similar to the preceding ones. Letting $N = (E \times F) \setminus M$,
we introduce the semimarkov kernel $\tilde{Q}$ by

(3.22) $\tilde{Q}(x,a; dy,db; ds) = Q(x,dy,ds) \, H(x,a,s; db) e^{-\sigma(b-a)} \, 1_N(y,b)$.

Then, using a Markov renewal argument at T_1 as before, for $(x,a) \epsilon N$,

(3.23) $Q_M(x,a; D,B; t) = \int_{D \times [0,t]} Q(x,dy,ds) \int_B H(x,a,s; db) e^{-\sigma(b-a)} 1_M(y,b)$

$$+ \int_{E \times F \times [0,t]} \tilde{Q}(x,a; dy,db; ds) \, Q_M(y,b; D,B; t-s)$$

and

(3.24) $\quad Q_k(x,a; D,B; t) = \int\limits_{[0,t]} ds \cdot q(x,s) \, 1_D(x) \int\limits_{B} H(x,a,s; db) e^{-\sigma(b-a)} \, r_k(x,b)$

$$+ \int\limits_{E\times F\times[0,t]} \tilde{Q}(x,a; dy,db; ds) \, Q_k(y,b; D,B; t-s) \ .$$

For fixed D and B, each of these equations is a Markov renewal equation of the form $f = g + \tilde{Q} * f$. Therefore, by (A.14), each has a unique solution of the form $f = \tilde{R} * g$, where

$$(3.25) \qquad\qquad\qquad \tilde{R} = \sum_{n=0}^{\infty} \tilde{Q}^n$$

is the Markov renewal kernel corresponding to $\tilde{Q}$ and where $g(x,a,t)$ is the first term on the right of (3.23) or (3.24). So,

(3.26) $\quad Q_M(x,a; D,B; t) = \int\limits_{E\times F\times[0,t]} \tilde{R}(x,a; dy,db; ds) \cdot \int\limits_{D\times[0,t-s]} Q(y,dz,du)$

$$\cdot \int\limits_{E} H(y,b,u; dc) \, e^{-\sigma(c-b)} \, 1_M(z,c)$$

and

(3.27) $\quad Q_k(z,a; D,B; t) = \int\limits_{E\times F\times[0,t]} \tilde{R}(x,a; dy,db; ds) \int\limits_{[0,t-s]} du \cdot q(y,u) \, 1_D(y)$

$$\cdot \int\limits_{B} H(y,b,u; dc) \, e^{-\sigma(c-b)} \, r_k(y,c) \ .$$

APPENDIX

This is a rapid account of notions we need from Markov renewal theory.
We refer to ÇINLAR (1969) and (1975) for further details.

As before we write $\mathbf{R}_+$ for $[0,\infty)$ we let E be a measurable space (the
σ-algebra of measurable subsets of E will be implicit). In applications, E
is sometimes the environmental state space E and sometimes the space
$\hat{E} = E \times F$.

A semimarkov kernel on E is a family of numbers $Q(x,C)$ in $[0,1]$,
defined for each $x \varepsilon E$ and each measurable subset C of $E \times \mathbf{R}_+$, such that

i) $x \to Q(x,C)$ is a measurable function on E, and

ii) $C \to Q(x,C)$ is a measure on $E \times \mathbf{R}_+$ whose total mass is at most one.
When $C = A \times B$, a measurable rectangle, we write $Q(x,A,B)$ instead of
$Q(x,A\times B)$. Similarly, for the measure element at $y \varepsilon E$ and $t \varepsilon \mathbf{R}_+$, we write
$Q(x,dy,dt)$ instead of $Q(x,d(y,t))$.

Given a semimarkov kernel Q on E, its iterates Q^n are defined recursive-
ly via

$$(A.1) \qquad Q^0(x,A,B) = \begin{cases} 1 & \text{if } x \varepsilon A \quad \text{and} \quad 0 \varepsilon B , \\ \\ 0 & \text{otherwise,} \end{cases}$$

$$(A.2) \qquad Q^{n+1}(x,A,B) = \int_{E \times \mathbf{R}_+} Q(x,dy,dt)\, Q^n(y,A,B-t) , \quad n \geq 0 ,$$

where $B - t = \{s - t : s \in B\}$. With these we define a kernel R by

$$(A.3) \qquad R(x,A,B) = \sum_{n=0}^{\infty} Q^n(x,A,B) \ ,$$

called the Markov renewal kernel corresponding to the kernel Q, which plays the same role in Markov renewal theory as the renewal function does in renewal theory.

Let Q be a semimarkov kernel on E and suppose that it is not defective, that is, $Q(x,E,\mathbb{R}_+) = 1$ for all x. A Markov renewal process with semimarkov kernel Q is a discrete parameter process (Y_n,T_n) taking values in $E \times \mathbb{R}_+$ such that

$$(A.4) \qquad P\left[Y_{n+1} \in A, \ T_{n+1} - T_n \in B \,|\, Y_0,\ldots,Y_n; \ T_0,\ldots,T_n\right] = Q(Y_n,A,B)$$

for all integers $n \geq 0$, all measurable $A \subset E$, and all Borel $B \subset \mathbb{R}_+$. Then, (Y_n) is a Markov chain with state space E and transition kernel

$$(A.5) \qquad P(x,A) = Q(x,A,\mathbb{R}_+) \ .$$

Also, it is easy to see that

$$(A.6) \qquad Q^n(x,A,B) = P\left[Y_n \in A, \ T_n \in B \,|\, Y_0 = x \ , \ T_0 = 0\right] \ ,$$

and that $R(x,A,B)$ is the expected number of times $Y_n \in A$ and $T_n \in B$, assuming that $Y_0 = x$ and $T_0 = 0$.

Associated with a Markov renewal process (Y_n, T_n) with $T_0 = 0$ and the added condition that

(A.7)
$$\lim_n T_n = \infty ,$$

we define a continuous parameter process (X_t) taking values in E by setting

(A.8)
$$X_t = Y_n \qquad \text{if} \qquad T_n \le t < T_{n+1} .$$

Note that X is piecewise constant, its jumps occur at $T_1, T_2, \ldots$, and the successive states it visits are $Y_0, Y_1, \ldots$. Such a process X is called a semimarkov process with state space F and semimarkov kernel Q.

In the applications here, we modeled the environment process X as a semimarkov process on the state space E with semimarkov kernel Q. In this case, the conditions that $Q(x, E, \mathbb{R}_+) = 1$ for all x and that $\lim T_n = +\infty$ are very natural. However, in subsequent computations, a number of other semimarkov kernels arose naturally but are defective: $\hat{Q}$ defined by (3.9), Q_M defined by (3.19), Q_k defined by (3.20), and $\hat{Q}$ defined by (3.22) are all semimarkov kernels on $\hat{E} = E \times F$ and are all defective, that is, for instance, $\hat{Q}(x, a; \hat{E}, \mathbb{R}_+) \le 1$ for most x. We do not associate Markov renewal processes with such defective kernels.

Markov renewal equations

Markov renewal equations are generalizations of renewal equations. We put here the essentials in the general case and give a result that got used a number of times in Section 3.

Let Q be a semimarkov kernel on E; we do allow it to be defective, that is, $Q(x,E,\mathbb{R}_+) < 1$ is possible. For a bounded measurable function f on $E \times \mathbb{R}_+$, we define the function $Q * f$ on $E \times \mathbb{R}_+$ by

$$(A.9) \qquad Q * f(x,t) = \int_{E \times [0,t]} Q(x,dy,ds) \, f(y,t-s) \,,$$

and define $Q * f$ and $R * f$ similarly (the latter is well-defined for f positive but may fail to exist for arbitrary bounded f).

Let f and g be bounded positive measurable functions on $E \times \mathbb{R}_+$. Suppose that

$$(A.10) \qquad f = g + Q * f \,.$$

This is called a Markov renewal equation. The following shows how to solve it (theoretically) and points out its strong resemblance to renewal equations.

(A.11) PROPOSITION. Suppose that $Q^n(x,E,[0,t]) \to 0$ as $n \to \infty$ for all $x \in E$ and $t \in \mathbb{R}_+$. Let f and g be bounded positive measurable functions satisfying (A.10). Then,

$$(A.12) \qquad f = R * g \,.$$

PROOF. We start by observing that $Q^n * (Q * f) = Q * (Q^n * f) = Q^{n+1} * f$, which follows from (A.9) and (A.2). Now, replacing f on the right side of (A.10) by $g + Q * f$, and repeating this operation, we obtain

$$f = g + Q * f = (g + Q * g + Q^2 * g + \ldots + Q^n * g) + Q^{n+1} * f$$

for any integer $n \geq 1$. In view of (A.3), the first term on the right increases to $R * g$, and we see that

(A.13)
$$f = R * g + \lim_n Q^{n+1} * f .$$

Fix $x \in E$ and $t \in \mathbb{R}_+$, and let f be bounded by the number b. Then,

$$Q^n * f(x,t) = \int_{E \times [0,t]} Q^n(x,dy,ds) \, f(y,t-s)$$

$$\leq b \int_{E \times [0,t]} Q^n(x,dy,ds) = b \, Q^n(x,E,[0,t]) ,$$

which approaches zero as $n \to \infty$ by hypothesis. Thus, $\lim Q^n * f = 0$ and in view of (A.13) the proof is complete.

The condition of the preceding proposition, namely, that $Q^n(x,E,[0,t]) \to 0$ as $n \to \infty$, is difficult to check in practice, especially when Q is given by a complicated expression as $\hat{Q}$ and $\tilde{Q}$ are. The following simplifies the task immensely. We state it in the form needed.

(A.14) PROPOSITION. Let Q be a non-defective semimarkov kernel on E and

suppose that the corresponding Markov renewal process (Y_n, T_n) is such that

$\lim T_n = + \infty$. Let $\hat{Q}$ be a semimarkov kernel on $\hat{E} = E \times F$ such that

(A.15) $\hat{Q}(x,a;\ dy,F;\ ds) \leq Q(x,dy,ds)$

for all $x \in E$, $a \in F$, $y \in E$, $s \in \mathbb{R}_+$. Then,

(A.16) $\lim\limits_n \hat{Q}^n(x,a;\ E,F;\ [0,t]) = 0$.

Therefore if f and g are bounded positive functions on $E \times F \times \mathbb{R}_+$ satisfying

$f = g + \hat{Q} * f$, then $f = \hat{R} * g$, where $\hat{R}$ is the Markov renewal kernel cor-

responding to $\hat{Q}$.

REMARK. The meaning of (A.15) is as follows: for any positive measurable

function f on $E \times \mathbb{R}_+$, the integral of f with respect to the kernel on the left-

side is less than or equal to the integral of f with respect to the kernel on

the right.

PROOF. Suppose that we have shown that

(A.17) $\hat{Q}^n(x,a;\ E,F;\ [0,t]) \leq Q^n(x,E,\ [0,t])$.

Then, since

$$Q^n(x, E, [0,t]) = P[T_n \leq t | Y_n = x] ,$$

and since $T_n \to \infty$ as $n \to \infty$ by hypothesis, we see that

$$\lim Q^n(x, E, [0,t]) = 0 ,$$

which in turn implies (A.16) via (A.17). the last statement of the proposition is now a restatement of Proposition (A.11), with $\hat{Q}$ replacing Q.

To complete the proof, then, we need to show (A.17). We do this by induction on n. For $n = 1$, (A.17) is immediate from the condition (A.15). *Suppose (A.17) holds for n.* Then,

$$\hat{Q}^{n+1}(x,a; E,F; [0,t])$$

$$= \int_{E \times F \times [0,t]} \hat{Q}(x,a; dy,db; ds) \, \hat{Q}^n(y,b; E,F; [0,t-s])$$

$$\leq \int_{E \times F \times [0,t]} \hat{Q}(x,a; dy,db; ds) \, Q^n(y,E,[0,t-s]) =$$

$$\leq \int_{E \times [0,t]} \hat{Q}(x,a; dy,F; ds) \, Q^n(y,E,[0,t-s]) \leq$$

$$\leq \int_{E \times [0,t]} Q(x,dy,ds) \, Q^n(y,E,[0,t]) = Q^{n+1}(x,E,[0,t]) .$$

Here, the first inequality is justified by the induction hypothesis, and the last inequality by the condition (A.15). So, (A.17) holds for n + 1 as well, and this completes the proof.

We remark that all the conditions of the preceding proposition hold when Q is the semimarkov kernel associated with the environment process and $\hat{Q}$ is either the $\hat{Q}$ of (3.9) or the $\tilde{Q}$ of (3.22).

REFERENCES

M. S. A-HAMEED and F. PROSCHAN (1973). Non-stationary shock models.
    Stoch. Process. Appl. 1, 383-404.

_____ and _____ (1975). Shock models with underlying birth process.
    J. Appl. Probab. 12, 18-28.

E. ÇINLAR (1969). On semi-Markov processes on arbitrary spaces.
    Proc. Cambridge Phil. Soc. 66, 381-392.

_____ (1975). Introduction to Stochastic Processes. Prentice-Hall,
    Englewood-Cliffs.

_____ (1977). Shock and wear models and Markov additive processes.
    In The Theory and Applications of Reliability, vol. 1,
    ed. by I. N. Shimi and C. P. Tsokos. Academic Press, New York.

_____ (1984). Markov and semimarkov models of deterioration.
    In Reliability Theory and Models, ed. by M. S. Abdel-Hameed,
    E. Çinlar, and J. Quinn. Academic Press, Orlando, Fla.

J. D. ESARY and A. W. MARSHALL (1974). Families of components and systems
    exposed to a compound Poisson damage process. In Reliability and
    Biometry, ed. by F. Proschan and R. J. Serfling. SIAM, Philadelphia.

_____, _____, and F. PROSCHAN (1973). Shock models and wear processes.
    Ann. Probab. 1, 627-649.

R. M. FELDMAN (1976). Optimal replacement for semimarkov shock models.
    J. Appl. Probab. 13, 195-199.

_____ (1977). Optimal replacement for systems governed by Markov additive
    shock processes. Ann. Probab. 5, 413-429.

D. V. LINDLEY and N. D. SINGPURWALLA (1984). Multivariate distributions
    for the reliability of a system of components sharing a common
    environment. Tech. Report, George Washington University,
    Institute of Reliability and Risk Analysis.

D. V. MASTRAN and N. D. SINGPURWALLA (1978). A Bayesian estimation of the
    reliability of coherent structures. Oper. Res. 26, 663-672.

H. PARKUS and H. BARGMAN (1970). Note on the behavior of thermorheologically
    simple materials in random temperature fields. Acta Mechanica 9,
    152-257.

M. SHAKED (1977). A concept of positive dependence for exchangeable
random variables. Ann. Statist. 5, 505-515.

A. WINTERBOTTOM (1984). The interval estimation of system reliability
from component test data. Oper. Res. 32, 628-640.

E. ÇINLAR   S. ÖZEKİCİ
Department of Civil Engineering   Department of Industrial Engineering
Princeton University   Bosphorus University
Princeton, New Jersey 08544   Bebek, Istanbul
U.S.A.   TURKEY

END

1-87

DTIC